Solitary waves in systems with separated Bragg grating and nonlinearity

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The existence and stability of solitons in a dual-core optical waveguide, in which one core has Kerr nonlinearity while the other one is linear with a Bragg grating written on it, are investigated. The system's spectrum for the frequency ω of linear waves always contains a gap. If the group velocity c in the linear core is zero, it also has two other, upper and lower (in terms of ω) gaps. If $c\neq 0$, the upper and lower gaps do not exist in the rigorous sense, as each overlaps with one branch of the continuous spectrum. When $c=0$, a family of zero-velocity soliton solutions, filling all the three gaps, is found analytically. Their stability is tested numerically, leading to a conclusion that only solitons in an upper section of the upper gap are stable. For $c\neq 0$, soliton solutions are sought for numerically. Stationary solutions are only found in the upper gap, in the form of unusual solitons, which exist as a continuous family in the former upper gap, *despite* its overlapping with one branch of the continuous spectrum. A region in the parameter plane (c,ω) is identified where these solitons are stable; it is again an upper section of the upper gap. Stable moving solitons are found too. A feasible explanation for the (virtual) existence of these solitons, based on an analytical estimate of their radiative-decay rate (if the decay takes place), is presented.

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I. INTRODUCTION

Gap solitary waves (which we will call solitons without implying integrability) exist in nonlinear dispersive media whose spectrum contains one or more forbidden bands, inside which linear waves cannot exist, giving room to exponentially localized states. An important example of an optical system with such a spectrum is a fiber equipped with a Bragg grating (BG). The interplay of Kerr nonlinearity and strong BG-induced dispersion gives rise to a vast family of *BG solitons* [1]. Comprehensive theoretical studies of these solitons have laid the ground for their experimental observations reported in Refs. [2]. Stable gap solitons have also been predicted in media combining BG with more sophisticated nonlinearities, such as quadratic [3] or that provided by narrow layers of resonantly absorbing two-level atoms, whose spacing is equal to the BG period [4]. More recently, it has been shown [5–7] that families of BG solitons can be made essentially more diverse in dual-core fibers. In particular, dual-core systems can be made *semilinear* [7] so that only one core is nonlinear, which gives rise to specific soliton dynamics [6,7].

In the systems considered so far, including the semilinear ones, nonlinearity and BG were presented in the same core. The objective of this work is to introduce and analyze a semilinear system where the nonlinearity and BG are physically separated, being placed in different cores. Although it may seem that a difference from the previously considered models amounts to technical details, we will demonstrate that, in fact, the spectrum of this system for the frequency ω of linear waves is drastically different from spectra of earlier considered models: it contains a central (in terms of ω) true gap, and two additional, lower and upper (mutually symmetric), ones. These are genuine gaps in the case when the group velocity c in the linear core is zero; otherwise, the upper and lower gaps each overlaps with one branch of the continuous spectrum. Generally speaking, gaps overlapping with the continuous spectrum are no longer gaps at all. Nevertheless, we will demonstrate that the (former) upper gap is a spectral band that contains a family of solitons (a part of which is stable) *despite* the overlapping with the continuous spectrum.

In the special case $c=0$, a family of exact soliton solutions filling all the three gaps is found in an analytical form. Numerical simulations demonstrate that, in this case, only the solitons in an upper section of the upper gap are stable. For $c\neq 0$, solitons are sought for by means of numerical methods, as no analytical solution is available in this case. As a result, *no* stationary solutions have been found in the genuine central gap (where they might be expected), nor in the lower gap, which gets immersed into the continuous spectrum. On the other hand, stationary solitons are found in the upper gap, (which is also immersed into the continuous spectrum if $c\neq 0$). They form (up to the accuracy of numerical computations) a continuous family inside the gap, and in a gap's upper section, they are found to be true stable solitons in any practical sense. This seemingly new type of solitons should be distinguished from recently identified *embedded solitons*, which also exist inside the linear spectrum, but they are isolated (discrete) *semi*-stable solutions that never form continuous families [8].

II. THE MODEL AND ITS LINEAR SPECTRUM

Following Ref. [7], it is straightforward to derive a model describing two linearly coupled cores, one with the Kerr nonlinearity and the other one equipped with BG,

$$iu_t+iu_x+[|v|^2+(1/2)|u|^2]u+\phi=0, \tag{1}$$

$$iv_t-iv_x+[|u|^2+(1/2)|v|^2]v+\psi=0, \tag{2}$$

$$i\phi_t+ic\phi_x+u+\lambda\psi=0, \tag{3}$$

$$i\psi_t-ic\psi_x+v+\lambda\phi=0. \tag{4}$$

Here, u and v are the forward- and backward-propagating waves in the nonlinear core, ϕ and ψ are their counterparts in the linear core, the coefficient of the linear coupling between the cores is normalized to be 1, and λ is the coefficient of linear coupling between the left- and right-propagating waves induced by BG in the linear core; it is always possible to set $\lambda > 0$. The group velocity in the nonlinear core is set equal to 1, and c is the group velocity in the linear core, measured in the same units. The two group velocities may be different, as the two cores can be made of different materials. As usual, the model neglects intrinsic dispersion (second-derivative terms) in both cores, since the effective dispersion induced by BG is much stronger [1] (it is easy to check that this remains true even in the case when BG is written on a single core of a dual-core system).

First, we consider the system's linear spectrum. Looking for a solution to the linearized equations in the form $u,v,\phi,\psi \sim \exp(ikx - i\omega t)$, we arrive at a dispersion relation

$$\omega^4 - [(1+c^2)k^2 + (2+\lambda^2)]\omega^2 + (ck^2-1)^2 + \lambda^2 k^2 = 0. \tag{5}$$

In the limiting case $c=0$, Eq. (5) yields three disjoint gaps. If $\lambda > 1/\sqrt{2}$, they are

$$\lambda < \omega < \sqrt{1+\lambda^2/4} + \lambda/2; \tag{6}$$

$$-(\sqrt{1+\lambda^2/4} - \lambda/2) < \omega < \sqrt{1+\lambda^2/4} - \lambda/2; \tag{7}$$

$$-(\sqrt{1+\lambda^2/4} + \lambda/2) < \omega < -\lambda, \tag{8}$$

and, if $\lambda < 1/\sqrt{2}$, the gaps are

$$\sqrt{1+\lambda^2/4} - \lambda/2 < \omega < \sqrt{1+\lambda^2/4} + \lambda/2; \tag{9}$$

$$-\lambda < \omega < \lambda; \tag{10}$$

$$-(\sqrt{1+\lambda^2/4} + \lambda/2) < \omega < -(\sqrt{1+\lambda^2/4} - \lambda/2). \tag{11}$$

Note that the gaps are symmetric with respect to the change of the sign of ω. In the particular case $\lambda = 1/\sqrt{2}$, all the three gaps merge into a single one, $-\sqrt{2} < \omega < \sqrt{2}$.

The character of the gaps becomes drastically different if $c \neq 0$. Indeed, in the case $c=0$ and $\lambda > 1/\sqrt{2}$ it is easy to see that Eq. (5) gives rise to two mutually symmetric branches $\omega(k)$ that start at the upper and lower edges of the central gap at $k=0$, and, monotonically varying, in the limit $|k| \to \infty$ they asymptotically approach constant values, which exactly coincide, respectively, with the lower edge of the upper gap and the upper edge of the lower gap [the dashed curves in the central part of Fig. 1(a)]. In the case $c=0$ and $\lambda < 1/\sqrt{2}$, the curves start at $k=0$ at the lower and upper edges of the upper and lower gaps, respectively, and asymptotically approach the upper and lower edges of the central gap at $|k| \to \infty$ [the dashed curves in the central part of Fig. 1(b)]. In either case, $\omega^2(|k|=\infty) = \lambda^2$. However, as it follows from Eq. (5), at any $c \neq 0$ the asymptotic form of these branches at $|k| \to \infty$ is totally different,

$$\omega^2 \approx \min\{c^2, 1\} k^2. \tag{12}$$

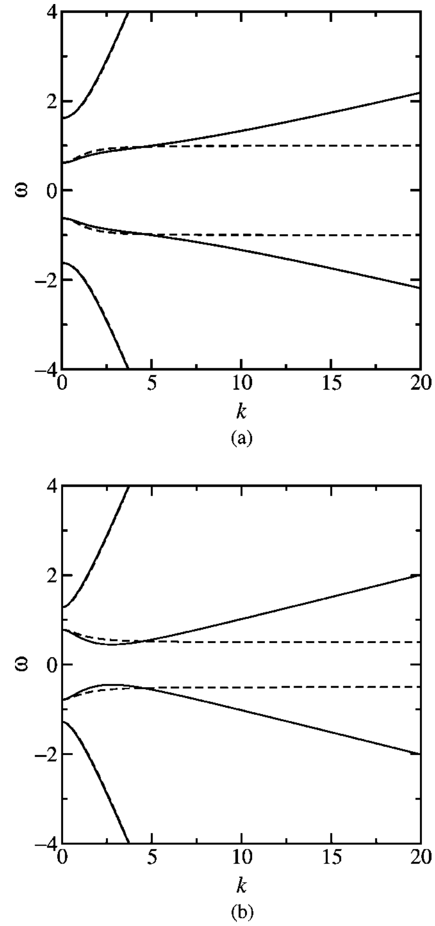

FIG. 1. Typical examples of the linear spectrum generated by the linearized equations (1)–(4): (a) $\lambda=1$, $c=0.1$; (b) $\lambda=1/2$, $c=0.1$. Dashed curves show the spectrum in the same systems with $c=0$.

Typical examples of the spectrum in the case of small c are displayed in Fig. 1. Evidently, the change of the shape of these dispersion curves gives rise to overlap of both the upper and lower gap with one (inner, in terms of Fig. 1) branch of the continuous spectrum, while the outer dispersion curves remain outside the gaps. As is suggested by what is known about the aforementioned embedded solitons [8], a gap which is overlapped with one branch, but continues to exist as a gap relative to another branch of the dispersion relation, may still be capable to support solitons of a special type, which will be seen below.

III. SOLITONS IN THE CASE $c=0$

In the case $c=0$, stationary soliton solutions to Eqs. (1)–(4) are sought for as

$$u = \eta U(x)\exp(-i\omega t), \quad v = \eta V(x)\exp(-i\omega t), \tag{13}$$

$$\phi = \eta \Phi(x)\exp(-i\omega t), \quad \psi = \eta \Psi(x)\exp(-i\omega t), \tag{14}$$

where the functions U,V, and Φ, Ψ are complex, and an extra constant η is introduced for convenience, see below. Substituting these expressions into Eqs. (1)–(4), it is possible to find exact solutions, following the pattern of the well-known generalized *Thirring solitons* in the single-core nonlinear fiber equipped with a BG [11],

$$U(x)=\sqrt{2/3}(\sin\theta)\ \operatorname{sech}(\eta^2 x\sin\theta - i\theta/2),$$
$$V(x)=-\sqrt{2/3}(\sin\theta)\operatorname{sech}(\eta^2 x\sin\theta + i\theta/2), \qquad (15)$$

$$\Phi=-\frac{\omega}{\omega^2-\lambda^2}U+\frac{\lambda}{\omega^2-\lambda^2}V,$$

$$\Psi=\frac{\lambda}{\omega^2-\lambda^2}U-\frac{\omega}{\omega^2-\lambda^2}V, \qquad (16)$$

where $\eta=\sqrt{\lambda/(\omega^2-\lambda^2)}$, θ is a real parameter that takes values $0<\theta<\pi$, and the frequency ω is to be found from a cubic equation

$$(\omega/\lambda)(\omega^2-\lambda^2-1)=\cos\theta. \qquad (17)$$

This equation yields three roots for ω at a given θ, one in each gap (recall that there are three genuine gaps in the case $c=0$). In particular, it is easy to check that the values $\theta=0$ and $\theta=\pi$, at which the soliton's amplitude vanishes according to Eqs. (15) and (16), exactly correspond to edges of the gaps (6)–(8) or (9)–(11): $\theta=0$ yields the upper edge of the upper gap, lower edge of the central gap, and upper edge of the lower gap, and $\theta=\pi$ gives rise to three other edge points of the gaps. Note that, although the soliton solutions (15), (16) completely fill all the three gaps (6)–(8) or (9)–(11), the solitons, unlike the gaps in which they exist, have no symmetry relative to the change of the sign of ω. The same will be true for solitons considered for $c\neq 0$ in the next section.

Stability of these exact soliton solutions was studied by means of direct simulations, which has yielded the following results: *all* the solitons belonging to the lower and central gaps are unstable, while the upper gap contains two sections, the solitons being stable in the upper section and unstable in the lower one. For instance, in the case $\lambda=1$, the unstable and stable sections inside the upper gap (6) are, respectively,

$$1<\omega<\omega_{\rm cr}\approx 1.525 \ \text{ and } \ \omega_{\rm cr}<\omega<(1/2)(\sqrt{5}+1)\approx 1.618, \qquad (18)$$

i.e., the stable section occupies $\approx 15\%$ of the upper gap.

These results are qualitatively consistent with those for the above-mentioned generalized Thirring solitons in the single-core fiber equipped with BG. In the latter case, the soliton family may be directly parametrized by the frequency, which takes values $-1<\omega<+1$. As was first demonstrated by means of variational approximation [9], and then by accurate numerical computations of stability eigenvalues [10], the generalized Thirring solitons are stable in an interval $\omega_{\rm cr}<\omega<1$, and unstable if $-1<\omega<\omega_{\rm cr}$, where a numerically found value $\omega_{\rm cr}$ is ≈ -0.02 (the value of $\omega_{\rm cr}$ predicted by means of the variational approximation in Ref. [9] is quite close to this). In that case, the (also upper) stable part of the single gap existing in the model occupies $\approx 50\%$ of the whole gap. The large difference in the relative size of the stable section in the model considered in the present work and in the above-mentioned generalized Thirring model shows that, although the models are qualitatively similar, their actual properties are essentially different.

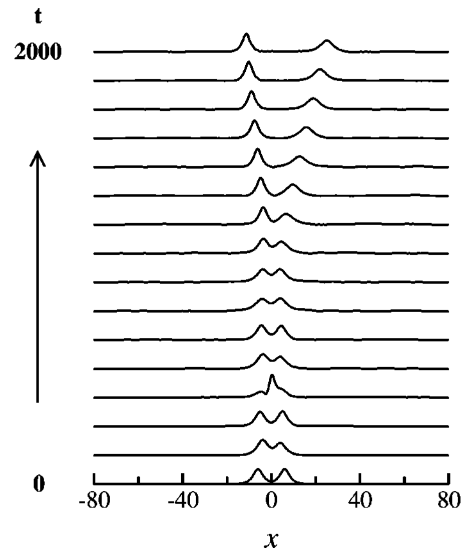

FIG. 2. Interaction between two identical stable solitons in the case $c=0$, $\lambda=1$, $\theta=\pi/4$ (corresponding to $\omega=1.565\,76$) and $\Delta\varphi=0$. Only the u component is shown.

We have also simulated interactions between two identical stable solitons, with an initial phase difference $\Delta\varphi$ between them. If $\Delta\varphi$ is zero, the solitons attract each other, and their collision generates moving solitons with *unequal* amplitudes and different velocities, see Fig. 2. This collision-induced spontaneous symmetry breaking may arise from the fact that a "lump," which is temporarily formed when the two solitons merge, is subject to modulational instability, so that small random perturbations can strongly distort it. In the cases $\Delta\varphi=\pi/2$ and $\Delta\varphi=\pi$, the solitons are found to repel each other. The symmetry breaking, as a result of the interaction between the solitons, occurs in these cases as well, but it is less conspicuous, especially when $\Delta\varphi=\pi$.

An implication of these results is that stable *moving* solitons also exist in the model. It should be noted that exact analytical solutions for moving solitons are known in the generalized Thirring model describing the single-core fiber carrying BG [11]), and moving solitons have been observed experimentally in such a fiber [2].

IV. THE CASE $c\neq 0$

In the general case, $c\neq 0$, no exact analytical solution is available. Stationary solutions can, however, be sought for numerically by solving equations produced by the substitution of the general expressions (13)–(14) into Eqs. (1)–(4). This numerical analysis has produced a surprising result: in the central gap, which remains a genuine one at $c\neq 0$ (see Fig. 1), *no* stationary solitons can be found, nor did we find any solution in the former lower gap. On the other hand, solitons are found in what was the upper gap at $c=0$. As it was explained above, the lower and upper gaps are immersed, each into one branch of the continuous spectrum, at

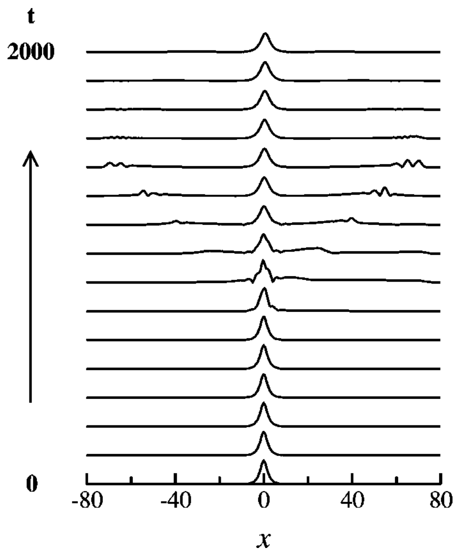

FIG. 3. Evolution of an unstable soliton with $c=0.2$, $\omega=1.5$, $\lambda=1$. Only the u component is shown.

any $c\neq 0$, that is why these solitons are unusual objects deserving detailed investigation.

To test the dynamical stability of these solitons, we simulated Eqs. (1)–(4), using the numerically obtained stationary solitons as initial conditions. As a result, it was found that they may be both stable and unstable. An example shown in Fig. 3 illustrates a general conclusion following from the simulations: if a soliton is unstable, it does not completely decay into radiation. Instead, in all the cases simulated, the unstable soliton sheds off some radiation and rearranges itself into a stable soliton with larger ω, larger width, and smaller amplitude. Thus, the stable solitons, although they occupy only a small part of the upper gap (see Fig. 4), appear to be strong *attractors* in the present model (in conservative nonlinear-wave models, attractors may exist due to radiative losses).

The results of the numerical stability analysis are summarized (for $\lambda=1$) in Fig. 4 in the form of a stability diagram on the (c,ω) plane. The right boundary, $c=c_{\max}(\omega)$ (shown by dots), separates stable and unstable solitons (we stress that the solitons of the present type, as stationary solutions to Eqs. (1)–(4), exist on both sides of the dotted boundary). The top horizontal line, which was found to coincide with the upper edge of the upper gap

$$\omega=\omega_{\mathrm{up}}\equiv\sqrt{1+\lambda^2/4}+\lambda/2 \tag{19}$$

[see Eqs. (6) and (9)], limits, as a matter of fact, the existence (rather than stability) region of the solitons. We have also investigated the situation at other values of λ, obtaining quite similar results. In particular, the stability region is smaller for smaller values of λ.

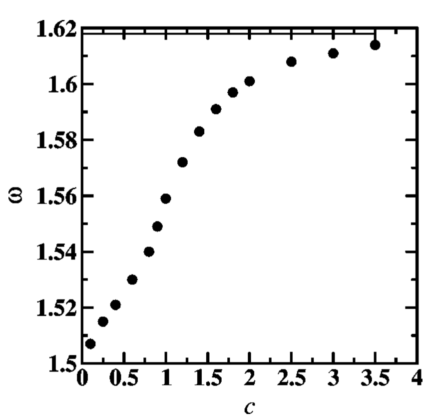

FIG. 4. The stability diagram in the plane (c,ω) at $\lambda=1$. Stable solitons overlapping with the continuous spectrum occupy the triangular region between the vertical axis, dotted curve, which is the border between stable and unstable solitons, and the horizontal line $\omega=(1/2)(\sqrt{5}+1)\approx 1.618$, which is the upper border of the gap in which the solitons exist.

Figure 4 strongly suggests that $c_{\max}(\omega)\rightarrow\infty$ as $\omega\rightarrow\omega_{\mathrm{up}}$. This feature can be readily understood, as well as the fact that the upper existence boundary (19) for the solitons does not depend on c. Indeed, recall that, in the case $c=0$, the upper edge of the upper gap (which is a genuine gap in that case), $\omega=\omega_{\mathrm{up}}$, exactly corresponds to $\theta=0$ according to Eq. (17). It is easy to check that, close to this edge, the width of the exact soliton solution (15), (16) diverges as $(\eta^2\sin\theta)^{-1}\sim(\omega_{\mathrm{up}}-\omega)^{-1/2}$, and its amplitude vanishes as $|\eta|\sin\theta\sim\sqrt{\omega_{\mathrm{up}}-\omega}$. With the diverging width, the x-derivative terms in Eqs. (3) and (4) become negligible [on the contrary to Eqs. (1) and (2), where the small derivatives are necessary to balance the small nonlinear terms]. Moreover, in this case it is straightforward to expand the stationary soliton solution in powers of the small parameter $c\sqrt{\omega_{\mathrm{up}}-\omega}$; we do not display the result as it is cumbersome. This explains why the upper edge of the upper gap does not depend on c, which is a coefficient in front of the terms that vanish exactly at the upper edge. This argument also supports the above conjecture that $c_{\max}(\omega)\rightarrow\infty$ as $\omega\rightarrow\omega_{\mathrm{up}}$.

A question may arise as to whether one may identify the exact (in a numerical sense) lower border $\omega_{\min}(c)$ of the region where the solitons exist for $c\neq 0$. As c approaches zero, $\omega_{\min}(c)$ approaches the lower edge of the true upper gap existing at $c=0$, see Fig. 1. However, for larger values of c, we did not aim to identify the lower boundary of the existence region of the solitons with a high accuracy because convergence of the numerical procedure deteriorates as one approaches the border. In any case, this lower border lies within that part of the soliton existence region where they are definitely unstable, therefore, it is not a feature of significant physical interest.

V. AN ESTIMATE FOR THE RATE OF RADIATIVE DECAY FOR THE SOLITONS OVERLAPPING WITH THE CONTINUOUS SPECTRUM

Due to the finite accuracy of the numerical methods, there still remains a fundamental question as to whether a continuous family of the solitons considered here exists in a rigorous sense, or the solitons would eventually decay into radiation, because of the resonance with the branch of the continuous spectrum with which they overlap, if the simulations could be run indefinitely long. If the latter is true, it is possible to estimate the corresponding soliton's decay rate. Following a perturbative formalism for the description of the energy emission by solitons coupled to the continuous spectrum [12], the energy P emitted by the soliton per unit of time (i.e., the emission power) is proportional to a squared integral of the following type:

$$P\sim\left|\int_{-\infty}^{+\infty}e^{ikx}u_{\mathrm{sol}}(x,t)\right|^{2}, \tag{20}$$

where k is the wave number of a linear wave coupled to the frequency ω of the soliton by the dispersion relation (5), and $u_{\mathrm{sol}}(x,t)$ is the soliton's wave field. For an estimate (which is definitely valid in the case of small c, when the perturbation theory is most relevant), we may use the asymptotic approximation (12), i.e., $k\approx\pm\omega/c$, and approximate the soliton by a simple wave form, $u(x,t)\sim\mathrm{sech}(x/W)\exp(-i\omega t)$, W being its characteristic half width. Substituting these approximations into Eq. (20), we obtain an exponential factor that determines the order of magnitude of the emission rate

$$P\sim\exp(-\pi W\omega/c), \tag{21}$$

(recall that the present solitons exist only with $\omega>0$). Then, for example in the case shown in Fig. 3, substitution of values of the parameters for the apparently stable final soliton into Eq. (21) yields $P\sim10^{-61}$, i.e., in this case the soliton may be regarded as a genuine one in any sense. Even if $c\gtrsim1$, the exponential factor remains extremely small. For instance, in a typical case of a stable soliton with $c=1.8$ and $\omega=1.6$, we find $W\sim10$, and $P\sim\exp(-28)\approx10^{-12}$.

In this connection, we stress that these estimates, concerning the *existence* of the solitons overlapping into the continuous spectrum, pertain equally to both stable and unstable solitons. The distinction between them (similar to the distinction between stable and unstable generalized Thirring solitons [9,10]) is a dynamical feature, absolutely different from their existence/nonexistence property.

Lastly, we have also simulated interactions between two identical stable solitons with a phase difference $\Delta\varphi$. In general, the results are similar to those briefly described above for the case $c=0$ (see Fig. 2, for instance). In particular, stable *moving* solitons of the present type exist too. For both cases $\Delta\varphi=0$ and $\Delta\varphi=\pi$, we observed that the interaction could additionally destabilize the solitons that were very close to the stability border.

VI. CONCLUSION

In this work, we have introduced a model of a dual-core optical system where one core has the Kerr nonlinearity and the other one is linear, being equipped with a Bragg grating. The linear spectrum of the system has a central gap, which is always a genuine one, and lower and upper gaps, each overlapping with one branch of the continuous spectrum, except for the case when the group velocity c in the linear core is zero. In the latter case, all the three gaps are genuine ones, and a family of soliton solutions is found in an exact form. These solutions completely fill all the three gaps, but only in an upper section of the upper one they are found to be dynamically stable. At $c\neq0$, the model gives rise to what seems to be a new type of solitons. In this case, no solitons are found in the genuine central gap and in the former lower one. On the other hand, in the upper gap, which overlaps with the continuous spectrum, unusual solitons were found. They exist as a continuous family inside the gap, *despite* its overlapping with one branch of the continuous spectrum. An upper section of the upper gap, in which these solitons are stable, has been identified. It was also found that the stable solitons may be set into motion as a result of their interaction, the moving solitons remaining stable.

Note added in proof. Very recently, A.R. Champneys has numerically investigated the characteristics of the stationary solutions by reducing the model to a set of ODEs and solving them using a very accurate numerical scheme based on the AUTO software package. As a result, it has been found that, strictly speaking, there is a dense but discrete system of embedded solitons in the region where our numerical results indicated the existence of the continuous family. However, as one is approaching the upper border of the region (see Fig. 4), the difference between truly localized embedded solitons and delocalized ones (with tiny spatially oscillating tails attached to them), existing in the gaps between the embedded solitons, becomes so small that no distinction between them is visible, and the soliton family is indeed getting continuous in any practical sense.

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